The Target Set Selection Problem on Cycle Permutation Graphs, Generalized Petersen Graphs and Torus Cordalis

Chun-Ying Chiang*, Liang-Hao Huang†, Wei-Ting Huang, Hong-Gwa Yeh‡§

Department of Mathematics, National Central University, Taiwan

Abstract

In this paper we consider a fundamental problem in the area of viral marketing, called Target Set Selection problem. In a a viral marketing setting, social networks are modeled by graphs with potential customers of a new product as vertices and friend relationships as edges, where each vertex v is assigned a threshold value $\theta(v)$. The thresholds represent the different latent tendencies of customers (vertices) to buy the new product when their friend (neighbors) do. Consider a repetitive process on social network (G, θ) where each vertex v is associated with two states, active and inactive, which indicate whether v is persuaded into buying the new product. Suppose we are given a target set $S \subseteq V(G)$. Initially, all vertices in G are inactive. At time step 0, we choose all vertices in S to become active. Then, at every time step $t > 0$, all vertices that were active in time step $t-1$ remain active, and we activate any vertex v if at least $\theta(v)$ of its neighbors were active at time step $t-1$. The activation process terminates when no more vertices can get activated. We are interested in the following optimization problem, called Target Set Selection: Finding a target set S of smallest possible size that activates all vertices of G. There is an important and well-studied threshold called strict majority threshold, where for every vertex v in G we have $\theta(v) = \lceil (d(v)+1)/2 \rceil$ and $d(v)$ is the degree of v in G. In this paper, we consider the Target Set Selection problem under strict majority thresholds and focus on three popular regular network structures: cycle permutation graphs, generalized Petersen graphs and torus cordalis.

Key words: Social networks, viral marketing, influence spreading, majority voting, dynamos, target set selection, irreversible k-threshold, majority threshold, cycle permutation graph, generalized Petersen graph, torus cordalis, tori.

*Partially supported by National Science Council under grant NSC97-2628-M-008-018-MY3.

†Partially supported by National Science Council under grant NSC98-2811-M-008-072.

‡Partially supported by National Science Council under grant NSC100-2811-M-008-052

§Corresponding author (hgyeh@math.ncu.edu.tw)

1 Introduction and preliminary results

A *graph* G consists of a set $V(G)$ of *vertices* together with a set $E(G)$ of unordered pairs of vertices called *edges*. We often use uv for an edge $\{u, v\}$. Two vertices u and v are *adjacent* to each other if $uv \in E(G)$.

In a viral marketing setting, a social network (G, θ) is a connected graph G equipped with thresholds $\theta : V(G) \to \mathbb{Z}$, where each vertex represents a potential customer of a new product, and each edge indicates that the two people are friends. The thresholds represent the different latent tendencies of vertices (customers) to buy the new product when their neighbors (friends) do. There are three types of important and well-studied thresholds called *k-constant threshold*, *majority threshold* and *strict majority threshold*. In a k-constant threshold, we have $\theta(v) = k$ for all vertices v of G, and (G, θ) is abbreviated to (G, k). In a majority threshold for every vertex v in G we have $\theta(v) = \lceil d(v)/2 \rceil$, while in a strict majority threshold we have $\theta(v) = \lceil (d(v) + 1)/2 \rceil$, where $d(v)$ is the degree of v in G.

In a social network (G, θ), every vertex in G has its own color which is either black or white, where black vertices represent *active* vertices, and white vertices represent *inactive* vertices. Given a set $S \subseteq V(G)$, consider the following repetitive process on (G, θ) called *activation process in (G, θ) starting at target set S*. Initially (at time 0), set all vertices in S to be black (with all other vertices white). After that, at each time step, the states of vertices are updated according to the following rule:

Parallel updating rule: All inactive vertices v that have at least $\theta(v)$ already-active neighbors become active.

The activation process terminates when no more vertices can get activated. The set of vertices that are active at the end of the process is denoted by $[S]^G_\theta$. If $F \subseteq [S]^G_\theta$, then we say that the target set S *influences* F in (G, θ). We are interested in the following optimization problem, called Target Set Selection: Finding a target set S of smallest possible size that influences all vertices in the social network (G, θ), that is $[S]^G_\theta = V(G)$ (such set S is called a *minimum seed* or an *optimal target set* for (G, θ)). We define $\text{min-seed}(G, \theta) = \min\{|S| : S \subseteq V(G) \text{ and } [S]^G_\theta = V(G)\}$.

The Target Set Selection problem and some of its variants were introduced and studied in [6, 9, 10, 16, 17, 20, 21, 22, 23, 24]. It is not surprising that Target Set Selection is NP-complete in general. Peleg [21] proved that it is NP-hard to compute the optimal target set for majority thresholds. In k-constant threshold setting, Dreyer and Roberts [10] showed that it is NP-hard to compute the $\text{min-seed}(G, k)$ for any $k \geq 3$, and Chen [6] showed that it is also NP-hard to compute $\text{min-seed}(G, 2)$. More surprising is the fact that $\text{min-seed}(G, \theta)$ is extremely hard to approximate. For any

graph G with majority thresholds θ, Chen [6] proved that min-seed(G, θ) cannot be approximated within the ratio $O(2^{\log^{1-\epsilon} n})$ for any fixed constant $\epsilon > 0$, unless $NP \subseteq DTIME(n^{polylog(n)})$, where $n = |V(G)|$.

Very little is known about the exact value of min-seed(G, θ). Related results can be found in [1, 2, 3, 6, 7, 10, 12, 13, 14, 15, 18, 19, 20, 21], where min-seed(G, θ) has been investigated under different threshold models for different types of network structure G: bounded treewidth graphs, trees, block-cactus graphs, chordal graphs, Hamming graphs, chordal rings, tori, meshes, butterflies. In the current paper, we consider Target Set Selection problem under strict majority thresholds and focus on three popular network structures: cycle permutation graphs, generalized Petersen graphs and torus cordalis.

Consider two identical disjoint copies G_1 and G_2 of a graph G with p vertices ($p \geq 4$), such that $V(G_1) = \{v_1, v_2, \ldots, v_p\}$ and $V(G_2) = \{u_1, u_2, \ldots, u_p\}$, where v_i and u_i are corresponding vertices for each i. For a permutation π on $\{1, 2, \ldots, p\}$, the π-*permutation graph* of G, denoted by $P_\pi(G)$, consists of G_1 and G_2 along with p additional edges $v_i u_{\pi(i)}$, $i = 1, 2, \ldots, p$. Note that the graph $P_\pi(G)$ depends not only on the choice of the permutation π but on the particular labeling of G as well. Permutation graphs were introduced in [4, 5]. The *n-path* P_n is the graph having $V(P_n) = \{x_1, x_2, \ldots, x_n\}$ and $E(P_n) = \{x_1x_2, x_2x_3, \ldots, x_{n-1}x_n\}$. The *n-cycle* C_n is the graph having $V(C_n) = V(P_n)$ and $E(C_n) = E(P_n) \cup \{x_nx_1\}$. If G is a cycle, then $P_\pi(G)$ is also called a *cycle permutation graph*. As examples, cycle permutation graphs $P_\pi(C_5)$ are depicted in Figure 1.

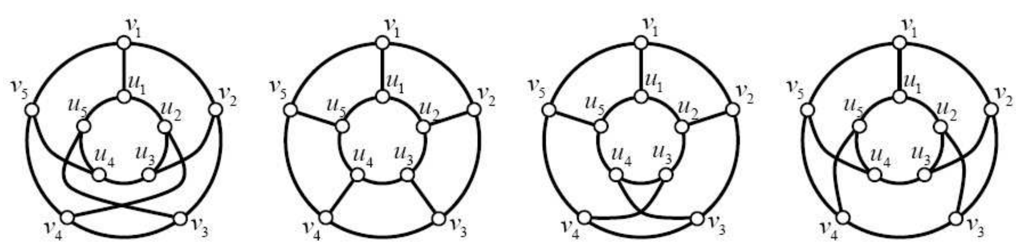

Figure 1. There are four cycle permutation graphs for $P_\pi(C_5)$.

For $m \geq 3$ and $1 \leq s \leq \lfloor \frac{m-1}{2} \rfloor$, the *generalized Petersen graph* $P(m, s)$ is defined to be the graph with vertex set

$$V(P(m, s)) = \{v_1, v_2, \ldots, v_m, u_1, u_2, \ldots, u_m\}$$

and edge set

$$E(P(m, s)) = \{v_iv_{i+1}, u_iv_i, u_iu_{i+s} : i = 1, 2, \ldots, m\},$$

where the subscripts are read modulo m. These graphs were introduced by Coxeter [8] and named by Watkins [26]. As examples, generalized Petersen graphs $P(5,2)$, $P(10,2)$, and $P(10,4)$ are depicted in Figure 2. The connection between generalized Petersen graphs and cycle permutation graphs is given in [25]. By the results in [25], we see that $P(10,4)$ is not a cycle permutation graph. Clearly, there are two cycle permutation graphs in Figure 1 which are not generalized Petersen graphs.

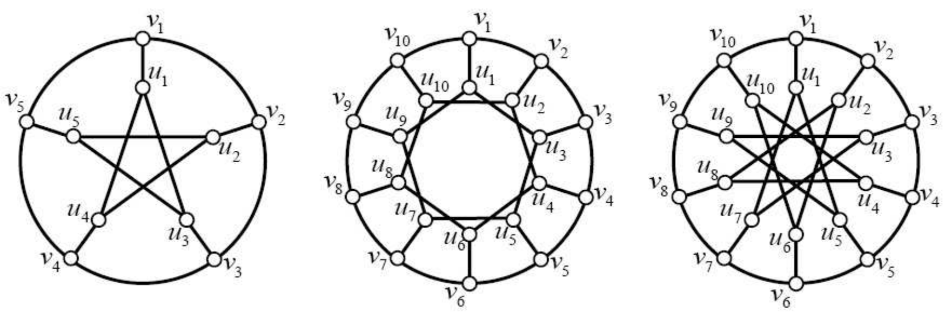

Figure 2. $P(5,2) \cong$ Petersen graph (left), $P(10,2) \cong$ dodecahedral graph (middle), and $P(10,4)$ (right).

The $m \times n$ *toroidal mesh* $C_m \Box C_n$ is the graph with vertex set $\{(i,j) : 1 \le i \le m, 1 \le j \le n\}$, where the neighbors of (i,j) are $(i-1,j), (i+1,j), (i,j-1), (i,j+1)$. Here the arithmetic in the first coordinate is modulo m and in the second coordinate modulo n. The $m \times n$ *torus cordalis* $C_m \oslash C_n$ and $m \times n$ toroidal mesh $C_m \Box C_n$ have the same vertex set. The edge set of $C_m \oslash C_n$ is almost the same as $C_m \Box C_n$, except that the edge $(i,n)(i,1)$ is replaced by the edge $(i,n)(i+1,1)$ for $1 \le i \le m$. The $m \times n$ *torus serpentinus* $C_m \otimes C_n$ is almost the same as $C_m \oslash C_n$, except that the edge $(1,j)(m,j)$ is replaced by the edge $(1,j)(m,j+1)$ for $1 \le j \le n$. As examples, $C_4 \Box C_3$, $C_4 \oslash C_3$ and $C_4 \otimes C_3$ are depicted in Figure 3.

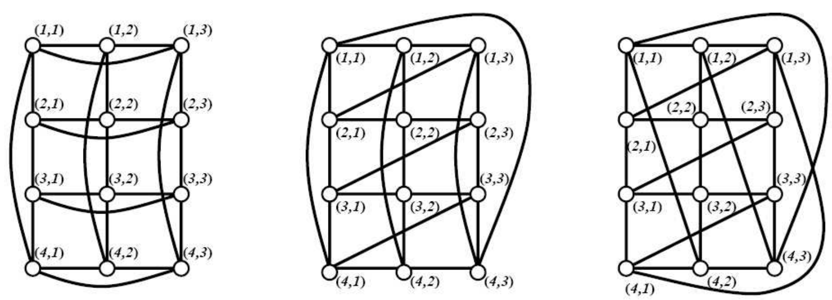

Figure 3. $C_4 \Box C_3$ (left), $C_4 \oslash C_3$ (middle), and $C_4 \otimes C_3$ (right).

In order to study the minimum seeds for (G,θ) we introduce a sequential version of activation process in (G,θ), called *sequential activation process*, which employs the following rule instead of the parallel updating rule:

Sequential updating rule: Exactly one of inactive vertices that have at least $\theta(v)$ already-active neighbors becomes active.

The proof of the following lemma is straightforward and so is omitted. In the sequel, Lemma 1 will be used without explicit reference to it.

Lemma 1 *A minimum seed for (G, θ) under sequential updating rule is also a minimum seed for (G, θ) under parallel updating rule, and vice versa.*

Consider a sequential activation process on (G, θ) starting from a target set S. In this process, if $v_1, v_2, \ldots, v_r$ is the order that vertices in $[S]_\theta^G \setminus S$ become black, then $[v_1, v_2, \ldots, v_r]$ is called the *convinced sequence* of S on (G, θ). In order to describe a convinced sequence, we introduce an operation $\sqcup$ on convinced subsequences. Let $\alpha = [v_1, v_2, \ldots, v_r]$ and $\beta = [u_1, u_2, \ldots, u_s]$. Then $\alpha \sqcup \beta$ is defined as

$$\alpha \sqcup \beta = [v_1, v_2, \ldots, v_r, u_1, u_2, \ldots, u_s],$$

and for a list of convinced subsequences $\{\alpha_{i,j}\}_{1 \le i \le \ell, 1 \le j \le k}$, the sequences $\sqcup_{i=1}^k \alpha_{i,j}$ and $\sqcup_{j=1}^\ell \sqcup_{i=1}^k \alpha_{i,j}$ are defined to be

$$\sqcup_{i=1}^k \alpha_{i,j} = \alpha_{1,j} \sqcup \alpha_{2,j} \sqcup \cdots \sqcup \alpha_{k,j} \text{ and } \sqcup_{j=1}^\ell \sqcup_{i=1}^k \alpha_{i,j} = \sqcup_{j=1}^\ell (\sqcup_{i=1}^k \alpha_{i,j}).$$

In Section 2, we precisely determine an optimal target set for a social network (G, θ) when G is a cycle permutation graph, and when G is a generalized Petersen graph. In [14], Flocchini *et al.* constructed the following bounds on the size of a minimum seed for toroidal mesh $C_m \Box C_n$, torus cordalis $C_m \oslash C_n$, and torus serpentinus $C_m \otimes C_n$ under strict majority thresholds.

Theorem 2 ([14]) *(a)* $\lceil \frac{mn+1}{3} \rceil \le \text{min-seed}(C_m \oslash C_n, 3) \le \lceil \frac{m}{3} \rceil (n+1)$. *(b) If G is a $C_m \Box C_n$ or a $C_m \otimes C_n$, then* $\lceil \frac{mn+1}{3} \rceil \le \text{min-seed}(G, 3) \le \min\{\lceil \frac{m}{3} \rceil (n+1), \lceil \frac{n}{3} \rceil (m+1)\}$.

In Section 3 of this paper, we present some improved upper bounds and exact values for the parameter min-seed$(C_m \oslash C_n, 3)$. These results are summarized in Table 1.

Theorems	$m \ge 10$	$n \ge 6$	Φ
Theorem 9(a)	odd	$n \equiv 0 \pmod 3$	$\Phi = \frac{mn}{3} + 1$
Theorem 9(b)	even	$n \equiv 0 \pmod 6$	$\Phi = \frac{mn}{3} + 1$
Theorem 9(c)	even	$n \equiv 3 \pmod 6$	$\Phi \in \{\frac{mn}{3} + 1, \frac{mn}{3} + 2\}$
Theorem 10		$n \equiv 1 \pmod 3$	$\Phi \le \frac{mn}{3} + \frac{m}{6} + 1$
Theorem 11		$n \equiv 2 \pmod 3$	$\Phi \le \frac{mn}{3} + \frac{m}{12} + \frac{3}{2}$
Theorem 12	$m \equiv 0 \pmod 3$		$\Phi = \frac{mn}{3} + 1$

Table 1. New bounds and exact values for min-seed$(C_m \oslash C_n, 3)$, where Φ denotes the parameter min-seed$(C_m \oslash C_n, 3)$.

2 Cycle permutation graphs and generalized Petersen graphs

For $X \subseteq V(G)$, let $G[X]$ denote the induced subgraph of G whose vertex set is X and whose edge set consists of all edges of G which have both ends in X. The number $\frac{1}{|V(G)|}\sum_{v\in V(G)} d(v)$ is defined to be the *average degree* of G and is denoted by $d(G)$. We remark that all lower bounds in Theorem 2 follows immediately from Lemma 4.

Theorem 3 ([11]) *Every graph with average degree at least $2k$, where k is a positive integer, has an induced subgraph with minimum degree at least $k+1$.*

Lemma 4 *Let G be a graph with n vertices, m edges and maximum degree Δ. If a target set $S \subseteq V(G)$ influences all vertices in the social network (G,k), then $|S| \geq \frac{m-(\Delta-k)n+1}{k}$.*

Proof. Let $\bar{S} = V(G)\setminus S$ and $G\setminus S = G[\bar{S}]$. Since S influences all vertices in the social network (G,k), the graph $G\setminus S$ has no induced subgraph with minimum degree at least $\Delta-k+1$. By Theorem 3, it follows that

$$2(\Delta-k) > d(G\setminus S) = \frac{2|E(G\setminus S)|}{|V(G\setminus S)|} \geq \frac{2(m-\Delta|S|)}{n-|S|},$$

where the last inequality follows from the fact that if e is an edge in $E(G)$ but not in $E(G\setminus S)$, then e has an end in S. We conclude that $(\Delta-k)(n-|S|) \geq m-\Delta|S|+1$ which completes the proof of the theorem. ∎

Theorem 5 *For any permutation π and $n \geq 4$,* $\text{min-seed}(P_\pi(C_n),2) = \lceil\frac{n+1}{2}\rceil$.

Proof. Let $G = P_\pi(C_n)$. Suppose that G consists of two disjoint copies G_1 and G_2 of C_n, such that $V(G_1) = \{v_1, v_2, \ldots, v_n\}$, $V(G_2) = \{u_1, u_2, \ldots, u_n\}$, $E(G_1) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1\}$, $E(G_2) = \{u_1u_2, u_2u_3, \ldots, u_{n-1}u_n, u_nu_1\}$ and $E(G) = E(G_1)\cup E(G_2)\cup \{v_iu_{\pi(i)} : i = 1,2,\ldots,n\}$. Without loss of generality, we might assume that $u_1v_1 \in E(G)$. Let $H = G[\{v_3, v_4, \ldots, v_n\}]$. In the proof of Proposition 2 of [10], it is shown that there is a minimum seed S' for the social network $(H,2)$ such that $|S'| = \lceil (n-1)/2\rceil$ and $\{v_3, v_n\} \subseteq S'$. Let $[v_{i_1}, v_{i_2}, \ldots, v_{i_a}]$ be the convinced sequence of S' on $(H,2)$, where $a = n-2-|S'|$.

Choose the target set $S = S' \cup \{u_1\}$ for $(G,2)$. It can easily be check that S can influence all vertices of $V(G)\setminus S$ in the social network $(G,2)$ by using $[v_{i_1}, v_{i_2}, \ldots, v_{i_a}] \sqcup [v_1, v_2] \sqcup [u_2, u_3, \ldots, u_n]$ as the convinced sequence. It follows that $\text{min-seed}(G,2) \leq$

$|S| = \lceil (n+1)/2 \rceil$. Moreover, from Lemma 4, it is easy to see that min-seed$(G, 2) \geq \lceil (n+1)/2 \rceil$. This completes the proof of the theorem. ∎

Clearly, if $\gcd(m, s) = 1$, then the generalized Petersen graph $P(m, s)$ is a cycle permutation graph, and we see at once that the following corollary holds. In Theorem 7 we further show that Corollary 6 holds if we drop the hypothesis $\gcd(m, s) = 1$.

Corollary 6 *If* $\gcd(m, s) = 1$, *then* min-seed$(P(m, s), 2) = \lceil \frac{m+1}{2} \rceil$.

Theorem 7 *For* $m \geq 3$ *and* $1 \leq s \leq \lfloor \frac{m-1}{2} \rfloor$, min-seed$(P(m, s), 2) = \lceil \frac{m+1}{2} \rceil$.

Proof. Let $G = P(m, s)$. Suppose that $V(G) = \{v_1, v_2, \ldots, v_m, u_1, u_2, \ldots, u_m\}$ and $E(G) = \{v_i v_{i+1}, u_i v_i, u_i u_{i+s} : i = 1, 2, \ldots, m\}$, where the subscripts are read modulo m. Let H be the graph $G[\{v_{s+1}, v_{s+2}, \ldots, v_{m-s}\}]$. Since H is a $(m - 2s)$-path, by the proof of Proposition 2 in [10], we get a minimum seed S' for the social network $(H, 2)$ such that $|S'| = \lceil (m - 2s + 1)/2 \rceil$ and $\{v_{s+1}, v_{m-s}\} \subseteq S'$. Let $[v_{j_1}, v_{j_2}, \ldots, v_{j_a}]$ be the convinced sequence of S' on $(H, 2)$, where $a = m - 2s - |S'|$.

Consider $S = S' \cup \{u_1, u_2, \ldots, u_s\}$ as a target set for $(G, 2)$. Since u_{m-s+i} is adjacent to both u_i and u_{m-2s+i} for $i \in \{1, 2, \ldots, s\}$ and u_{s+j} is adjacent to both u_j and v_{s+j} for $j \in \{1, 2, \ldots, m - 2s\}$, we see that S can influence all vertices of $V(G) \setminus S$ in the social network $(G, 2)$ by using $[v_{j_1}, v_{j_2}, \ldots, v_{j_a}] \sqcup [v_s, v_{s-1}, \ldots, v_1] \sqcup [u_{s+1}, u_{s+2}, \ldots, u_{m-s}] \sqcup [u_{m-s+1}, u_{m-s+2}, \ldots, u_m] \sqcup [v_{m-s+1}, v_{m-s+2}, \ldots, v_m]$ as the convinced sequence. Therefore min-seed$(G, 2) \leq |S| = |S'| + s = \lceil (m+1)/2 \rceil$. Moreover, by Lemma 4, it can be seen that min-seed$(G, 2) \geq \lceil (m+1)/2 \rceil$. This complete the proof of the theorem. ∎

3 Torus cordalis

Theorem 8 min-seed$(C_m \oslash C_3, 3) = m + 1$ *for any* $m \geq 3$.

Proof. Let $G = C_m \oslash C_3$. First we show that min-seed$(G, 3) \leq m + 1$ by giving a target set $S \subseteq V(G)$ which influences all vertices of G. Denote by S_1 and S_2 the sets $\{(2i+1, 1) : 0 \leq i \leq \lfloor \frac{m-1}{2} \rfloor\}$ and $\{(2i, 2) : 1 \leq i \leq \lfloor \frac{m}{2} \rfloor\}$, respectively. Set $S = S_1 \cup S_2 \cup \{(1, 3)\}$. Let $\alpha_1 = [(2, 1), (4, 1), \ldots, (2\lfloor \frac{m}{2} \rfloor, 1)]$, $\alpha_2 = [(1, 2), (3, 2), \ldots, (1 + 2\lfloor \frac{m-1}{2} \rfloor, 2)]$ and $\alpha_3 = [(2, 3), (3, 3), \ldots, (m, 3)]$. It is straightforward to see that $\alpha_1 \sqcup \alpha_2 \sqcup \alpha_3$ is a convinced sequence of S on $(G, 3)$ (see Figure 1 in Appendix for a graphical illustration of this convinced sequence). By the lower bound of min-seed$(C_m \oslash C_3, 3)$ in Theorem 2(a), we see that min-seed$(C_m \oslash C_3, 3) = m + 1$. ∎

Theorem 9 *Let $s \geq 2$ be an integer. (a) If $m \geq 5$ is an odd integer, then* min-seed$(C_m \oslash C_{3s}, 3) = ms + 1$. *(b) If $m \geq 8$ is an even integer and s is an even integer, then* min-seed$(C_m \oslash C_{3s}, 3) = ms + 1$. *(c) If $m \geq 8$ is an even integer and s is an odd integer, then* min-seed$(C_m \oslash C_{3s}, 3) = ms + 1$ *or* $ms + 2$.

Proof. Let $G = C_m \oslash C_{3s}$.

(a) Denote by S_1 and S_2 the sets $\cup_{j=0}^{s-1}\{(1, 1+3j), (2, 2+3j), (3, 1+3j), (4, 3+3j), (5, 2+3j)\}$ and $\cup_{j=0}^{s-1} \cup_{i=0}^{\frac{m-7}{2}} \{(6+2i, 3+3j), (7+2i, 2+3j)\}$, respectively. Let $S = S_1 \cup S_2 \cup \{(4, 1)\}$. In the social network $(G, 3)$, it is straightforward to check that the target set S can influence all vertices in $V(G) \setminus S$ by using the convinced sequence $\alpha = \alpha_1 \sqcup \alpha_2 \sqcup \alpha_3 \sqcup \alpha_4 \sqcup \alpha_5$, where $\alpha_1 = \sqcup_{j=0}^{s-1}[(1, 2+3j), (2, 1+3j)]$, $\alpha_2 = \sqcup_{j=0}^{s-1} \sqcup_{i=0}^{\frac{m-7}{2}} [(5+2i, 3+3j), (6+2i, 2+3j)]$, $\alpha_3 = [(5, 1), (6, 1), (7, 1), \ldots, (m, 1)]$, $\alpha_4 = [(4, 2), (3, 2), (3, 3), (2, 3), (1, 3), (m, 3)]$, and $\alpha_5 = \sqcup_{j=0}^{s-2}([(m, 4+3j), (m-1, 4+3j), \ldots, (4, 4+3j)] \sqcup [(4, 5+3j), (3, 5+3j), (3, 6+3j), (2, 6+3j), (1, 6+3j), (m, 6+3j)])$ (see Figure 2 in Appendix for a graphical illustration of this convinced sequence α). Therefore min-seed$(C_m \oslash C_{3s}, 3) \leq |S| = ms + 1$ and hence by Theorem 2(a), we have min-seed$(C_m \oslash C_{3s}, 3) = ms + 1$.

(b) Denote by S_1, S_2 and S_3 the sets $\cup_{j=0}^{s-1}\{(1, 1+3j), (2, 2+3j), (3, 1+3j), (4, 3+3j), (5, 2+3j)\}$, $\cup_{j=0}^{s-1} \cup_{i=0}^{\frac{m-10}{2}} \{(6+2i, 3+3j), (7+2i, 2+3j)\}$ and $\cup_{j=0}^{s-1}\{(m-2, 1+3j), (m-1, 3+3j), (m, 2+3j)\}$, respectively. Let $S = S_1 \cup S_2 \cup S_3 \cup \{(4, 1)\}$. It can readily be checked that the target set S can influence all vertices in $V(G) \setminus S$ by using the convinced sequence $\alpha = \alpha_1 \sqcup \alpha_2 \sqcup \alpha_3 \sqcup \alpha_4 \sqcup \alpha_5 \sqcup \alpha_6 \sqcup \alpha_7$ (see Figure 3 in Appendix for a graphical illustration of this convinced sequence α), where

$$\alpha_1 = \sqcup_{j=0}^{s-1}[(1, 2+3j), (2, 1+3j)],$$

$$\alpha_2 = \sqcup_{j=0}^{s-1} \sqcup_{i=0}^{\frac{m-10}{2}} [(5+2i, 3+3j), (6+2i, 2+3j)],$$

$$\alpha_3 = [(5, 1), (6, 1), (7, 1), \ldots, (m-3, 1)] \sqcup [(m-3, 3s)],$$

$$\alpha_4 = [(4, 2), (3, 2), (3, 3), (2, 3), (1, 3), (m, 3)],$$

$$\alpha_5 = \sqcup_{k=0}^{\frac{s-4}{2}}([(m, 4+6k), (m-1, 4+6k), (m-1, 5+6k), (m-2, 5+6k), (m-2, 6+6k), (m-3, 6+6k)] \sqcup [(m-3, 7+6k), (m-4, 7+6k), \ldots, (4, 7+6k)] \sqcup [(4, 8+6k), (3, 8+6k), (3, 9+6k), (2, 9+6k), (1, 9+6k), (m, 9+6k)]),$$

$$\alpha_6 = [(m, 3s-2), (m-1, 3s-2), (m-1, 3s-1), (m-2, 3s-1), (m-2, 3s)], \text{ and}$$

$$\alpha_7 = \sqcup_{k=0}^{\frac{s-2}{2}}([(m, 1+6k), (m-1, 1+6k), (m-1, 2+6k), (m-2, 2+6k), (m-2, 3+6k), (m-3, 3+6k)] \sqcup [(m-3, 4+6k), (m-4, 4+6k), \ldots, (4, 4+6k)] \sqcup [(4, 5+6k), (3, 5+6k), (3, 6+6k), (2, 6+6k), (1, 6+6k), (m, 6+6k)]).$$

Therefore min-seed$(C_m \oslash C_{3s}, 3) \le |S| = ms+1$ and hence by Theorem 2(a), we have min-seed$(C_m \oslash C_{3s}, 3) = ms + 1$.

(c) Denote by S_1, S_2 and S_3 the sets $\cup_{j=0}^{s-1}\{(1,1+3j),(2,2+3j),(3,1+3j),(4,3+3j),(5,2+3j)\}$, $\cup_{j=0}^{s-1}\cup_{i=0}^{\frac{m-10}{2}}\{(6+2i,3+3j),(7+2i,2+3j)\}$ and $\cup_{j=0}^{s-1}\{(m-2,1+3j),(m-1,3+3j),(m,2+3j)\}$, respectively. Let $S = S_1 \cup S_2 \cup S_3 \cup \{(4,1),(m-1,1)\}$. It is straightforward to check that the target set S can influence all vertices in $V(G)\setminus S$ by using the convinced sequence $\alpha = \alpha_1 \sqcup \alpha_2 \sqcup \alpha_3 \sqcup \alpha_4 \sqcup \alpha_5 \sqcup \alpha_6 \sqcup \alpha_7$ (see Figure 4 in Appendix for a graphical illustration of this convinced sequence α), where

$\alpha_1 = \sqcup_{j=0}^{s-1}[(1,2+3j),(2,1+3j)]$,

$\alpha_2 = \sqcup_{j=0}^{s-1}\sqcup_{i=0}^{\frac{m-10}{2}}[(5+2i,3+3j),(6+2i,2+3j)]$,

$\alpha_3 = [(5,1),(6,1),(7,1),\ldots,(m-3,1)]$,

$\alpha_4 = [(4,2),(3,2),(3,3),(2,3),(1,3),(m,3)]$,

$\alpha_5 = [(m,1),(m-1,2),(m-2,2),(m-2,3),(m-3,3)]$,

$\alpha_6 = \sqcup_{k=0}^{\frac{s-3}{2}}([(m-3,4+6k),(m-4,4+6k),\ldots,(4,4+6k)]\sqcup[(4,5+6k),(3,5+6k),(3,6+6k),(2,6+6k),(1,6+6k),(m,6+6k)]\sqcup[(m,7+6k),(m-1,7+6k),(m-1,8+6k),(m-2,8+6k),(m-2,9+6k),(m-3,9+6k)])$, and

$\alpha_7 = \sqcup_{k=0}^{\frac{s-3}{2}}([(m,4+6k),(m-1,4+6k),(m-1,5+6k),(m-2,5+6k),(m-2,6+6k),(m-3,6+6k)]\sqcup[(m-3,7+6k),(m-4,7+6k),\ldots,(4,7+6k)]\sqcup[(4,8+6k),(3,8+6k),(3,9+6k),(2,9+6k),(1,9+6k),(m,9+6k)])$.

Therefore min-seed$(C_m \oslash C_{3s}, 3) \le |S| = ms+2$ and hence by Theorem 2(a), we have min-seed$(C_m \oslash C_{3s}, 3) = ms + 1$ or $ms + 2$. ∎

Theorem 10 *If* $m \ge 8$ *and* $n \equiv 1 \pmod 3$, *then* min-seed$(C_m \oslash C_n, 3) \le \frac{mn}{3} + \frac{m}{6} + 1$.

Proof. Let $G = C_m \oslash C_n$ and $n = 3s+1$. The proof is divided into three cases, according to the parity of the two integers m and s.

Case 1. $m \ge 5$ is odd. Let S_1, S_2 and S_3 denote the sets $\cup_{j=0}^{s-1}\{(1,2+3j),(2,3+3j),(3,2+3j),(4,4+3j),(5,3+3j)\}$, $\{(6,1),(8,1),\ldots,(m-1,1)\}$ and $\cup_{j=0}^{s-1}\cup_{i=0}^{\frac{m-7}{2}}\{(6+2i,4+3j),(7+2i,3+3j)\}$, respectively. Let $S = \{(1,1),(2,1),(4,1)\} \cup S_1 \cup S_2 \cup S_3$. It is straightforward to check that the target set S can influence all vertices in $V(G)\setminus S$ by using the convinced sequence $\alpha = \alpha_1 \sqcup \alpha_2 \sqcup \alpha_3 \sqcup \alpha_4$, where $\alpha_1 = \sqcup_{j=0}^{s-1}[(1,3+3j),(2,2+3j)]$, $\alpha_2 = \sqcup_{j=0}^{s-1}\sqcup_{i=0}^{\frac{m-7}{2}}[(5+2i,4+3j),(6+2i,3+3j)]$, $\alpha_3 =$

$[(3,1),(5,1),(7,1),\ldots,(m,1)]$, and $\alpha_4 = \sqcup_{j=0}^{s-1}([(m,2+3j),(m-1,2+3j),\ldots,(4,2+3j)] \sqcup [(4,3+3j),(3,3+3j),(3,4+3j),(2,4+3j),(1,4+3j),(m,4+3j)])$ (see Figure 5 in Appendix for a graphical illustration of this convinced sequence α). Therefore min-seed$(C_m \oslash C_n, 3) \le |S| = \frac{mn}{3} + \frac{m}{6} + \frac{1}{2}$.

Case 2. $m \ge 8$ is even and s is odd. Denote by S_1, S_2, S_3 and S_4 the sets $\cup_{j=0}^{s-1}\{(1,2+3j),(2,3+3j),(3,2+3j),(4,4+3j),(5,3+3j)\}$, $\{(6,1),(8,1),\ldots,(m-4,1)\}$, $\cup_{j=0}^{s-1}\cup_{i=0}^{\frac{m-10}{2}}\{(6+2i,4+3j),(7+2i,3+3j)\}$ and $\cup_{j=0}^{s-1}\{(m-2,2+3j),(m-1,4+3j),(m,3+3j)\}$, respectively. Let $S = S_1 \cup S_2 \cup S_3 \cup S_4 \cup \{(1,1),(2,1),(4,1),(m-1,1)\}$. It can readily be checked that the target set S can influence all vertices in $V(G) \setminus S$ by using the convinced sequence $\alpha = \alpha_1 \sqcup \alpha_2 \sqcup \alpha_3 \sqcup \alpha_4 \sqcup \alpha_5 \sqcup \alpha_6 \sqcup \alpha_7$ (see Figure 6 in Appendix for a graphical illustration of this convinced sequence α), where

$$\alpha_1 = \sqcup_{j=0}^{s-1}[(1,3+3j),(2,2+3j)],$$

$$\alpha_2 = \sqcup_{j=0}^{s-1}\sqcup_{i=0}^{\frac{m-10}{2}}[(5+2i,4+3j),(6+2i,3+3j)],$$

$$\alpha_3 = [(3,1),(5,1),(7,1),\ldots,(m-5,1)],$$

$$\alpha_4 = [(m,1),(m,2),(m-1,2),(m-1,3),(m-2,3),(m-2,4),(m-3,4)],$$

$$\alpha_5 = \sqcup_{k=0}^{\frac{s-3}{2}}([(m-3,5+6k),(m-4,5+6k),\ldots,(4,5+6k)] \sqcup [(4,6+6k),(3,6+6k),(3,7+6k),(2,7+6k),(1,7+6k),(m,7+6k)] \sqcup [(m,8+6k),(m-1,8+6k),(m-1,9+6k),(m-2,9+6k),(m-2,10+6k),(m-3,10+6k)]),$$

$$\alpha_6 = [(m-2,1),(m-3,1)] \sqcup [(m-3,2),(m-4,2),\ldots,(4,2)] \sqcup [(4,3),(3,3),(3,4),(2,4),(1,4),(m,4)], \text{ and}$$

$$\alpha_7 = \sqcup_{k=0}^{\frac{s-3}{2}}([(m,5+6k),(m-1,5+6k),(m-1,6+6k),(m-2,6+6k),(m-2,7+6k),(m-3,7+6k)] \sqcup [(m-3,8+6k),(m-4,8+6k),\ldots,(4,8+6k)] \sqcup [(4,9+6k),(3,9+6k),(3,10+6k),(2,10+6k),(1,10+6k),(m,10+6k)]).$$

Therefore min-seed$(C_m \oslash C_n, 3) \le |S| = \frac{mn}{3} + \frac{m}{6}$.

Case 3. $m \ge 8$ and s are both even. Denote by S_1, S_2, S_3 and S_4 the sets $\cup_{j=0}^{s-1}\{(1,2+3j),(2,3+3j),(3,2+3j),(4,4+3j),(5,3+3j)\}$, $\{(6,1),(8,1),\ldots,(m-4,1)\}$, $\cup_{j=0}^{s-1}\cup_{i=0}^{\frac{m-10}{2}}\{(6+2i,4+3j),(7+2i,3+3j)\}$ and $\cup_{j=0}^{s-1}\{(m-2,2+3j),(m-1,4+3j),(m,3+3j)\}$, respectively. Let $S = S_1 \cup S_2 \cup S_3 \cup S_4 \cup \{(1,1),(2,1),(4,1),(m-2,1),(m-1,1)\}$. It is straightforward to check that the target set S can influence all vertices in $V(G) \setminus S$ by using the convinced sequence $\alpha = \alpha_1 \sqcup \alpha_2 \sqcup \alpha_3 \sqcup \alpha_4 \sqcup \alpha_5$ (see Figure 7 in Appendix for a graphical illustration of this convinced sequence α), where

$\alpha_1 = \sqcup_{j=0}^{s-1}[(1, 3+3j), (2, 2+3j)]$,

$\alpha_2 = \sqcup_{j=0}^{s-1} \sqcup_{i=0}^{\frac{m-10}{2}} [(5+2i, 4+3j), (6+2i, 3+3j)]$,

$\alpha_3 = [(3, 1), (5, 1), (7, 1), \ldots, (m-3, 1)] \sqcup [(m, 1)]$,

$\alpha_4 = \sqcup_{k=0}^{\frac{s-2}{2}}([(m-3, 2+6k), (m-4, 2+6k), \ldots, (4, 2+6k)] \sqcup [(4, 3+6k), (3, 3+6k), (3, 4+6k), (2, 4+6k), (1, 4+6k), (m, 4+6k)] \sqcup [(m, 5+6k), (m-1, 5+6k), (m-1, 6+6k), (m-2, 6+6k), (m-2, 7+6k), (m-3, 7+6k)])$, and

$\alpha_5 = \sqcup_{k=0}^{\frac{s-2}{2}}([(m, 2+6k), (m-1, 2+6k), (m-1, 3+6k), (m-2, 3+6k), (m-2, 4+6k), (m-3, 4+6k)] \sqcup [(m-3, 5+6k), (m-4, 5+6k), \ldots, (4, 5+6k)] \sqcup [(4, 6+6k), (3, 6+6k), (3, 7+6k), (2, 7+6k), (1, 7+6k), (m, 7+6k)])$.

We conclude that min-seed$(C_m \oslash C_n, 3) \leq |S| = \frac{mn}{3} + \frac{m}{6} + 1$. This completes the proof of the theorem. ∎

Theorem 11 *If* $m \geq 10$, $n \equiv 2 \pmod 3$ *and* $n \geq 5$, *then* min-seed$(C_m \oslash C_n, 3) \leq \frac{mn}{3} + \frac{m}{12} + \frac{3}{2}$.

Proof. Let $G = C_m \oslash C_n$, $m = 4t + r$ and $n = 3s + 2$, where t, r, s are integers with $0 \leq r \leq 3$. The proof is divided into six cases, according to the value of r and the parity of s.

Case 1. $r = 0$ and s is even. In this case, let S_1, S_2, S_3 and S_4 denote the sets $\cup_{j=0}^{s-1}\{(1, 3+3j), (2, 4+3j), (3, 3+3j)\}$, $\cup_{i=0}^{t-3}\{(4+4i, 1), (6+4i, 1), (6+4i, 2)\}$, $\cup_{j=0}^{s-1}\cup_{i=0}^{t-3}\{(4+4i, 5+3j), (5+4i, 3+3j), (6+4i, 5+3j), (7+4i, 3+3j)\}$ and $\cup_{j=0}^{s-1}\{(m-4, 5+3j), (m-3, 4+3j), (m-2, 3+3j), (m-1, 5+3j), (m, 4+3j)\}$, respectively. Let $S = \{(2, 1), (3, 2), (m-4, 1), (m-3, 2), (m-1, 1), (m, 2)\} \cup S_1 \cup S_2 \cup S_3 \cup S_4$. It is straightforward to check that the target set S can influence all vertices in $V(G) \setminus S$ by using the convinced sequence $\alpha = \alpha_1 \sqcup \alpha_2 \sqcup \alpha_3 \sqcup \alpha_4 \sqcup \alpha_5 \sqcup \alpha_6 \sqcup \alpha_7 \sqcup \alpha_8$ (see Figure 8 in Appendix for a graphical illustration of this convinced sequence α), where

$\alpha_1 = \sqcup_{j=0}^{s-1}[(1, 4+3j), (2, 3+3j)]$,

$\alpha_2 = \sqcup_{i=0}^{t-3}[(7+4i, 1), (7+4i, 2), (5+4i, 1), (5+4i, 2), (4+4i, 2)]$,

$\alpha_3 = \sqcup_{j=0}^{s-1} \sqcup_{i=0}^{t-3} [(4+4i, 3+3j), (5+4i, 5+3j), (6+4i, 3+3j), (7+4i, 5+3j)]$,

$\alpha_4 = [(m-4, 2), (m-3, 1)]$,

$\alpha_5 = \sqcup_{k=0}^{\frac{s-2}{2}}([(m-3, 3+6k), (m-4, 3+6k)] \sqcup [(m-4, 4+6k), (m-5, 4+6k), (m-6, 4+6k), \ldots, (3, 4+6k)] \sqcup [(3, 5+6k), (2, 5+6k), (1, 5+6k), (m, 5+6k)] \sqcup [(m, 6+6k), (m-1, 6+6k), (m-1, 7+6k), (m-2, 7+6k), (m-2, 8+6k), (m-3, 8+6k)])$,

$\alpha_6 =$ $[(m-2,1),(m-2,2),(m-1,2),(m,1)]$,

$\alpha_7 =$ $[(3,1),(2,2),(1,2),(1,1)]$, and

$\alpha_8 =$ $\sqcup_{k=0}^{\frac{s-2}{2}}([(m,3+6k),(m-1,3+6k),(m-1,4+6k),(m-2,4+6k),(m-2,5+6k),(m-3,5+6k),(m-3,6+6k),(m-4,6+6k)]\sqcup[(m-4,7+6k),(m-5,7+6k),(m-6,7+6k),\ldots,(3,7+6k)]\sqcup[(3,8+6k),(2,8+6k),(1,8+6k),(m,8+6k)])$.

We conclude that min-seed$(C_m \oslash C_n, 3) \le |S| = \frac{mn}{3} + \frac{m}{12}$.

Case 2. $r = 0$ and s is odd. In this case, let S_1, S_2, S_3 and S_4 denote the sets $\cup_{j=0}^{s-1}\{(1,3+3j),(2,4+3j),(3,3+3j)\}$, $\cup_{i=0}^{t-3}\{(4+4i,1),(6+4i,1),(6+4i,2)\}$, $\cup_{j=0}^{s-1}\cup_{i=0}^{t-3}\{(4+4i,5+3j),(5+4i,3+3j),(6+4i,5+3j),(7+4i,3+3j)\}$ and $\cup_{j=0}^{s-1}\{(m-4,5+3j),(m-3,4+3j),(m-2,3+3j),(m-1,5+3j),(m,4+3j)\}$, respectively. Let $S = \{(2,1),(3,2),(m-4,1),(m-3,2),(m-2,1),(m-1,1),(m,2)\} \cup S_1 \cup S_2 \cup S_3 \cup S_4$. It is straightforward to check that the target set S can influence all vertices in $V(G)\backslash S$ by using the convinced sequence $\alpha = \alpha_1 \sqcup \alpha_2 \sqcup \alpha_3 \sqcup \alpha_4 \sqcup \alpha_5 \sqcup \alpha_6 \sqcup \alpha_7 \sqcup \alpha_8 \sqcup \alpha_9$ (see Figure 9 in Appendix for a graphical illustration of this convinced sequence α), where

$\alpha_1 =$ $\sqcup_{j=0}^{s-1}[(1,4+3j),(2,3+3j)]$,

$\alpha_2 =$ $\sqcup_{i=0}^{t-3}[(7+4i,1),(7+4i,2),(5+4i,1),(5+4i,2),(4+4i,2)]$,

$\alpha_3 =$ $\sqcup_{j=0}^{s-1}\sqcup_{i=0}^{t-3}[(4+4i,3+3j),(5+4i,5+3j),(6+4i,3+3j),(7+4i,5+3j)]$,

$\alpha_4 =$ $[(m-4,2),(m-3,1),(m-2,2),(m-1,2),(m,1)]$,

$\alpha_5 =$ $[(3,1),(2,2),(1,2),(1,1)]$,

$\alpha_6 =$ $[(m-3,3),(m-4,3)]\sqcup[(m-4,4),(m-5,4),(m-6,4),\ldots,(3,4)]\sqcup[(3,5),(2,5),(1,5),(m,5)]$,

$\alpha_7 =$ $[(m,3),(m-1,3),(m-1,4),(m-2,4),(m-2,5),(m-3,5)]$,

$\alpha_8 =$ $\sqcup_{k=0}^{\frac{s-3}{2}}([(m-3,6+6k),(m-4,6+6k)]\sqcup[(m-4,7+6k),(m-5,7+6k),(m-6,7+6k),\ldots,(3,7+6k)]\sqcup[(3,8+6k),(2,8+6k),(1,8+6k),(m,8+6k)]\sqcup[(m,9+6k),(m-1,9+6k),(m-1,10+6k),(m-2,10+6k),(m-2,11+6k),(m-3,11+6k)])$, and

$\alpha_9 =$ $\sqcup_{k=0}^{\frac{s-3}{2}}([(m,6+6k),(m-1,6+6k),(m-1,7+6k),(m-2,7+6k),(m-2,8+6k),(m-3,8+6k),(m-3,9+6k),(m-4,9+6k)]\sqcup[(m-4,10+6k),(m-5,10+6k),(m-6,10+6k),\ldots,(3,10+6k)]\sqcup[(3,11+6k),(2,11+6k),(1,11+6k),(m,11+6k)])$.

It follows that min-seed$(C_m \oslash C_n, 3) \leq |S| = \frac{mn}{3} + \frac{m}{12} + 1$.

Case 3. $r = 1$. In this case, let S_1, S_2, S_3 and S_4 denote the sets $\cup_{j=0}^{s-1}\{(1, 3+3j), (2, 4+3j), (3, 3+3j)\}$, $\cup_{i=0}^{t-2}\{(4+4i, 1), (6+4i, 1), (6+4i, 2)\}$, $\cup_{j=0}^{s-1}\cup_{i=0}^{t-2}\{(4+4i, 5+3j), (5+4i, 3+3j), (6+4i, 5+3j), (7+4i, 3+3j)\}$ and $\cup_{j=0}^{s-1}\{(m-1, 5+3j), (m, 4+3j)\}$, respectively. Let $S = \{(2, 1), (3, 2), (m-1, 1), (m, 2)\} \cup S_1 \cup S_2 \cup S_3 \cup S_4$. It is straightforward to check that the target set S can influence all vertices in $V(G) \setminus S$ by using the convinced sequence $\alpha = \alpha_1 \sqcup \alpha_2 \sqcup \alpha_3 \sqcup \alpha_4 \sqcup \alpha_5 \sqcup \alpha_6$ (see Figure 10 in Appendix for a graphical illustration of this convinced sequence α), where

$$\alpha_1 = \sqcup_{j=0}^{s-1}[(1, 4+3j), (2, 3+3j)],$$

$$\alpha_2 = \sqcup_{i=0}^{t-2}[(7+4i, 1), (7+4i, 2), (5+4i, 1), (5+4i, 2), (4+4i, 2)],$$

$$\alpha_3 = \sqcup_{j=0}^{s-1}\sqcup_{i=0}^{t-2}[(4+4i, 3+3j), (5+4i, 5+3j), (6+4i, 3+3j), (7+4i, 5+3j)],$$

$$\alpha_4 = [(m-1, 2), (m, 1)],$$

$$\alpha_5 = [(3, 1), (2, 2), (1, 2), (1, 1)], \text{ and}$$

$$\alpha_6 = \sqcup_{j=0}^{s-1}([(m, 3+3j), (m-1, 3+3j)] \sqcup [(m-1, 4+3j), (m-2, 4+3j), (m-3, 4+3j), \ldots, (3, 4+3j)] \sqcup [(3, 5+3j), (2, 5+3j), (1, 5+3j), (m, 5+3j)]).$$

Therefore min-seed$(C_m \oslash C_n, 3) \leq |S| = \frac{mn}{3} + \frac{m}{12} + \frac{1}{4}$.

Case 4. $r = 2$ and s is even. In this case, let S_1, S_2, S_3 and S_4 denote the sets $\cup_{j=0}^{s-1}\{(1, 3+3j), (2, 4+3j), (3, 3+3j), (4, 5+3j), (5, 3+3j)\}$, $\cup_{i=0}^{t-3}\{(6+4i, 1), (8+4i, 1), (8+4i, 2)\}$, $\cup_{j=0}^{s-1}\cup_{i=0}^{t-3}\{(6+4i, 5+3j), (7+4i, 3+3j), (8+4i, 5+3j), (9+4i, 3+3j)\}$ and $\cup_{j=0}^{s-1}\{(m-4, 5+3j), (m-3, 4+3j), (m-2, 3+3j), (m-1, 5+3j), (m, 4+3j)\}$, respectively. Let $S = \{(2, 1), (3, 2), (4, 1), (5, 2), (m-4, 1), (m-3, 2), (m-1, 1), (m, 2)\} \cup S_1 \cup S_2 \cup S_3 \cup S_4$. It can readily be checked that the target set S can influence all vertices in $V(G) \setminus S$ by using the convinced sequence $\alpha = \alpha_1 \sqcup \alpha_2 \sqcup \alpha_3 \sqcup \alpha_4 \sqcup \alpha_5 \sqcup \alpha_6 \sqcup \alpha_7 \sqcup \alpha_8 \sqcup \alpha_9 \sqcup \alpha_{10}$ (see Figure 11 in Appendix for a graphical illustration of this convinced sequence α), where

$$\alpha_1 = \sqcup_{j=0}^{s-1}[(1, 4+3j), (2, 3+3j)],$$

$$\alpha_2 = [(4, 2), (5, 1)],$$

$$\alpha_3 = \sqcup_{i=0}^{t-3}[(9+4i, 1), (9+4i, 2), (7+4i, 1), (7+4i, 2), (6+4i, 2)],$$

$$\alpha_4 = \sqcup_{j=0}^{s-1}[(4, 3+3j), (5, 5+3j)],$$

$$\alpha_5 = \sqcup_{j=0}^{s-1}\sqcup_{i=0}^{t-3}[(6+4i, 3+3j), (7+4i, 5+3j), (8+4i, 3+3j), (9+4i, 5+3j)],$$

$$\alpha_6 = [(m-4,2),(m-3,1)],$$

$$\alpha_7 = \sqcup_{k=0}^{\frac{s-2}{2}}([(m-3,3+6k),(m-4,3+6k)]\sqcup[(m-4,4+6k),(m-5,4+6k),(m-6,4+6k),\dots,(3,4+6k)]\sqcup[(3,5+6k),(2,5+6k),(1,5+6k),(m,5+6k)]\sqcup[(m,6+6k),(m-1,6+6k),(m-1,7+6k),(m-2,7+6k),(m-2,8+6k),(m-3,8+6k)]),$$

$$\alpha_8 = [(m-2,1),(m-2,2),(m-1,2),(m,1)],$$

$\alpha_9 = [(3,1),(2,2),(1,2),(1,1)]$, and

$$\alpha_{10} = \sqcup_{k=0}^{\frac{s-2}{2}}([(m,3+6k),(m-1,3+6k),(m-1,4+6k),(m-2,4+6k),(m-2,5+6k),(m-3,5+6k),(m-3,6+6k),(m-4,6+6k)]\sqcup[(m-4,7+6k),(m-5,7+6k),(m-6,7+6k),\dots,(3,7+6k)]\sqcup[(3,8+6k),(2,8+6k),(1,8+6k),(m,8+6k)]).$$

It follows that min-seed$(C_m \oslash C_n, 3) \leq |S| = \frac{mn}{3} + \frac{m}{12} + \frac{1}{2}$.

Case 5. $r = 2$ and s is odd. In this case, let S_1, S_2, S_3 and S_4 denote the sets $\cup_{j=0}^{s-1}\{(1,3+3j),(2,4+3j),(3,3+3j),(4,5+3j),(5,3+3j)\}$, $\cup_{i=0}^{t-3}\{(6+4i,1),(8+4i,1),(8+4i,2)\}$, $\cup_{j=0}^{s-1}\cup_{i=0}^{t-3}\{(6+4i,5+3j),(7+4i,3+3j),(8+4i,5+3j),(9+4i,3+3j)\}$ and $\cup_{j=0}^{s-1}\{(m-4,5+3j),(m-3,4+3j),(m-2,3+3j),(m-1,5+3j),(m,4+3j)\}$, respectively. Let $S = \{(2,1),(3,2),(4,1),(5,2),(m-4,1),(m-3,2),(m-2,1),(m-1,1),(m,2)\} \cup S_1 \cup S_2 \cup S_3 \cup S_4$. It is straightforward to check that the target set S can influence all vertices in $V(G) \setminus S$ by using the convinced sequence $\alpha = \alpha_1 \sqcup \alpha_2 \sqcup \alpha_3 \sqcup \alpha_4 \sqcup \alpha_5 \sqcup \alpha_6 \sqcup \alpha_7 \sqcup \alpha_8 \sqcup \alpha_9 \sqcup \alpha_{10} \sqcup \alpha_{11}$ (see Figure 12 in Appendix for a graphical illustration of this convinced sequence α), where

$$\alpha_1 = \sqcup_{j=0}^{s-1}[(1,4+3j),(2,3+3j)],$$

$$\alpha_2 = [(4,2),(5,1)],$$

$$\alpha_3 = \sqcup_{i=0}^{t-3}[(9+4i,1),(9+4i,2),(7+4i,1),(7+4i,2),(6+4i,2)],$$

$$\alpha_4 = \sqcup_{j=0}^{s-1}[(4,3+3j),(5,5+3j)],$$

$$\alpha_5 = \sqcup_{j=0}^{s-1}\sqcup_{i=0}^{t-3}[(6+4i,3+3j),(7+4i,5+3j),(8+4i,3+3j),(9+4i,5+3j)],$$

$$\alpha_6 = [(m-4,2),(m-3,1),(m-2,2),(m-1,2),(m,1)],$$

$$\alpha_7 = [(3,1),(2,2),(1,2),(1,1)],$$

$$\alpha_8 = [(m-3,3),(m-4,3)]\sqcup[(m-4,4),(m-5,4),(m-6,4),\dots,(3,4)]\sqcup[(3,5),(2,5),(1,5),(m,5)],$$

$$\alpha_9 = [(m,3),(m-1,3),(m-1,4),(m-2,4),(m-2,5),(m-3,5)],$$

$\alpha_{10} = \sqcup_{k=0}^{\frac{s-3}{2}}([(m-3, 6+6k), (m-4, 6+6k)] \sqcup [(m-4, 7+6k), (m-5, 7+6k), (m-6, 7+6k), \ldots, (3, 7+6k)] \sqcup [(3, 8+6k), (2, 8+6k), (1, 8+6k), (m, 8+6k)] \sqcup [(m, 9+6k), (m-1, 9+6k), (m-1, 10+6k), (m-2, 10+6k), (m-2, 11+6k), (m-3, 11+6k)])$, and

$\alpha_{11} = \sqcup_{k=0}^{\frac{s-3}{2}}([(m, 6+6k), (m-1, 6+6k), (m-1, 7+6k), (m-2, 7+6k), (m-2, 8+6k), (m-3, 8+6k), (m-3, 9+6k), (m-4, 9+6k)] \sqcup [(m-4, 10+6k), (m-5, 10+6k), (m-6, 10+6k), \ldots, (3, 10+6k)] \sqcup [(3, 11+6k), (2, 11+6k), (1, 11+6k), (m, 11+6k)])$.

We conclude that min-seed$(C_m \oslash C_n, 3) \leq |S| = \frac{mn}{3} + \frac{m}{12} + \frac{3}{2}$.

Case 6. $r = 3$. In this case, let S_1, S_2, S_3 and S_4 denote the sets $\cup_{j=0}^{s-1}\{(1, 3+3j), (2, 4+3j), (3, 3+3j), (4, 5+3j), (5, 3+3j)\}$, $\cup_{i=0}^{t-2}\{(6+4i, 1), (8+4i, 1), (8+4i, 2)\}$, $\cup_{j=0}^{s-1}\cup_{i=0}^{t-2}\{(6+4i, 5+3j), (7+4i, 3+3j), (8+4i, 5+3j), (9+4i, 3+3j)\}$ and $\cup_{j=0}^{s-1}\{(m-1, 5+3j), (m, 4+3j)\}$, respectively. Let $S = \{(2,1), (3,2), (4,1), (5,2), (m-1, 1), (m, 2)\} \cup S_1 \cup S_2 \cup S_3 \cup S_4$. It is straightforward to check that the target set S can influence all vertices in $V(G) \setminus S$ by using the convinced sequence $\alpha = \alpha_1 \sqcup \alpha_2 \sqcup \alpha_3 \sqcup \alpha_4 \sqcup \alpha_5 \sqcup \alpha_6 \sqcup \alpha_7 \sqcup \alpha_8$ (see Figure 13 in Appendix for a graphical illustration of this convinced sequence α), where

$\alpha_1 = \sqcup_{j=0}^{s-1}[(1, 4+3j), (2, 3+3j)]$,

$\alpha_2 = [(4, 2), (5, 1)]$,

$\alpha_3 = \sqcup_{i=0}^{t-2}[(9+4i, 1), (9+4i, 2), (7+4i, 1), (7+4i, 2), (6+4i, 2)]$,

$\alpha_4 = \sqcup_{j=0}^{s-1}[(4, 3+3j), (5, 5+3j)]$,

$\alpha_5 = \sqcup_{j=0}^{s-1}\sqcup_{i=0}^{t-2}[(6+4i, 3+3j), (7+4i, 5+3j), (8+4i, 3+3j), (9+4i, 5+3j)]$,

$\alpha_6 = [(m-1, 2), (m, 1)]$,

$\alpha_7 = [(3, 1), (2, 2), (1, 2), (1, 1)]$, and

$\alpha_8 = \sqcup_{j=0}^{s-1}([(m, 3+3j), (m-1, 3+3j)] \sqcup [(m-1, 4+3j), (m-2, 4+3j), (m-3, 4+3j), \ldots, (3, 4+3j)] \sqcup [(3, 5+3j), (2, 5+3j), (1, 5+3j), (m, 5+3j)])$.

Therefore min-seed$(C_m \oslash C_n, 3) \leq |S| = \frac{mn}{3} + \frac{m}{12} + \frac{3}{4}$. This completes the proof of the theorem. ∎

Theorem 12 *If* $m \equiv 0 \pmod 3$ *and* $n \geq 2$, *then* min-seed$(C_m \oslash C_n, 3) = \frac{mn}{3} + 1$.

Proof. Let $G = C_m \oslash C_n$ and $m = 3t$. The proof is divided into two cases, according to the parity of n.

Case 1. n is even. Denote by S_1, S_2 and S_3 the sets $\cup_{j=0}^{\frac{n-2}{2}}\{(1,1+2j),(2,2+2j)\}$, $\cup_{i=0}^{t-2}\{(4+3i,2),(6+3i,1)\}$ and $\cup_{j=0}^{\frac{n-4}{2}}\cup_{i=0}^{t-2}\{(4+3i,4+2j),(5+3i,3+2j)\}$, respectively. Let $S = S_1 \cup S_2 \cup S_3 \cup \{(3,1)\}$. It is straightforward to check that the target set S can influence all vertices in $V(G) \setminus S$ by using the convinced sequence $\alpha = \alpha_1 \sqcup \alpha_2 \sqcup \alpha_3 \sqcup \alpha_4$, where $\alpha_1 = \sqcup_{j=0}^{\frac{n-2}{2}}[(2,1+2j),(1,2+2j)]$, $\alpha_2 = \sqcup_{i=0}^{t-2}\sqcup_{j=0}^{\frac{n-4}{2}}[(4+3i,3+2j),(5+3i,4+2j)]$, $\alpha_3 = [(3,2),(3,3),(3,4),\ldots,(3,n)]$, and $\alpha_4 = \sqcup_{i=0}^{t-2}([(4+3i,1),(5+3i,1),(5+3i,2)] \sqcup [(6+3i,2),(6+3i,3),(6+3i,4),\ldots,(6+3i,n)])$ (see Figure 14 in Appendix for a graphical illustration of this convinced sequence α). Therefore min-seed$(C_m \oslash C_n, 3) \leq |S| = \frac{mn}{3} + 1$. By Theorem 2(a), we conclude that min-seed$(C_m \oslash C_n, 3) = \frac{mn}{3} + 1$.

Case 2. n is odd. By Theorem 8, it suffices to consider only the case when $n \geq 5$. Denote by S_1, S_2 and S_3 the sets $\cup_{i=0}^{t-1}\{(1+3i,1),(2+3i,2),(3+3i,3)\}$, $\cup_{j=0}^{\frac{n-5}{2}}\{(1,5+2j),(2,4+2j)\}$ and $\cup_{j=0}^{\frac{n-5}{2}}\cup_{i=0}^{t-2}\{(4+3i,4+2j),(5+3i,5+2j)\}$, respectively. Let $S = S_1 \cup S_2 \cup S_3 \cup \{(1,3)\}$. It is straightforward to check that the target set S can influence all vertices in $V(G) \setminus S$ by using the convinced sequence $\alpha = \alpha_1 \sqcup \alpha_2 \sqcup \alpha_3 \sqcup \alpha_4$ (see Figure 15 in Appendix for a graphical illustration of this convinced sequence α), where

$$\alpha_1 = \sqcup_{j=0}^{\frac{n-3}{2}}[(2,1+2j),(1,2+2j)],$$

$$\alpha_2 = \sqcup_{i=0}^{t-2}\sqcup_{j=0}^{\frac{n-7}{2}}[(4+3i,5+2j),(5+3i,6+2j)],$$

$$\alpha_3 = \sqcup_{i=0}^{t-2}([(m-3i,n),(m-3i,n-1),\ldots,(m-3i,4)] \sqcup [(m-1-3i,4),(m-1-3i,3),(m-2-3i,3),(m-2-3i,2),(m-3i,2),(m-3i,1),(m-1-3i,1),(m-2-3i,n)]), \text{ and}$$

$$\alpha_4 = [(3,2),(3,1),(2,n)] \sqcup [(3,n),(3,n-1),\ldots,(3,4)].$$

Therefore min-seed$(C_m \oslash C_n, 3) \leq |S| = \frac{mn}{3} + 1$, and hence by Theorem 2(a) we get min-seed$(C_m \oslash C_n, 3) = \frac{mn}{3} + 1$. This completes the proof of the theorem. ∎

References

[1] E. Ackerman, O. Ben-Zwi, G. Wolfovitz, *Combinatorial model and bounds for target set selection*, Theoretical Computer Science 411 (2010) 4017-4022.

[2] O. Ben-Zwi, D. Hermelin, D. Lokshtanov, I. Newman, *Treewidth governs the complexity of target set selection*, Discrete Optimization 8 (2011) 87-96.

[3] E. Berger, *Dynamic monopolies of constant size*, J. Combin. Theory Ser. B, 83 (2001) 191-200.

[4] G. Chartrand, F. Harary, *Planar Permutation Graphs*, Ann. Inst. Henri. Poincaré Sect. B 3 (1967) 433-438.

[5] G. Chartrand, J. B. Frechen, *On the Chromatic Number of Permutation Graphs.* In: F. Harary (ed.): Proof Techniques in Graph Theory, Proceedings of the Second Ann Arbor Graph Theory Conference, February 1968. Academic Press, New York, pages 21-24, 1969.

[6] N. Chen *On the approximability of influence in social networks*, SIAM J. Discrete Math. 23 (2009) 1400-1415.

[7] C-Y Chiang, L-H Huang, B-J Li, Jiaojiao Wu, Hong-Gwa Yeh, *Some Results on the Target Set Selection Problem*, submitted. Also in: arXiv:1111.6685v1.

[8] H. S. M. Coxeter, *Self-Dual Configurations and Regular Graphs*, Bull. Amer. Math. Soc. 56 (1950) 413-455.

[9] P. Domingos, M. Richardson, *Mining the network value of customers*, in Proceedings of ACM SIGKDD 2001, San Francisco, pages 57-66, 2001.

[10] P. A. Dreyer, F. S. Roberts, *Irreversible k-threshold processes: Graph-theoretical threshold models of the spread of disease and of opinion*, Discrete Applied Math. 157 (2009) 1615-1627.

[11] P. Erdős, *On an extremal problem in graph theory*, Colloq. Math. 13 (1964/1965) 251-254.

[12] P. Flocchini, F. Geurts, N. Santoro, *Optimal irreversible dynamos in chordal rings*, Discrete Applied Mathematics 113 (2001) 23-42.

[13] P. Flocchini, R. Královič, P. Ruźička, A. Roncato, N. Santoro. *On time versus size for monotone dynamic monopolies in regular topologies*, Journal of Discrete Algorithms 1 (2003) 129-150.

[14] P. Flocchini, E. Lodi, F. Luccio, L. Pagli, N. Santoro, *Dynamic monopolies in tori*, Discrete Applied Mathematics 137 (2004) 197-212.

[15] P. Flocchini, *Contamination and Decontamination in Majority-Based Systems*, Journal of Cellular Automata 4 (2009) 183-200.

[16] D. Kempe, J. Kleinberg, E. Tardos, *Maximizing the spread of influence through a social network*, in Proceedings of the 9th ACM SIGKDD international conference on Knowledge discovery and data mining, pages 137-146, 2003.

[17] D. Kempe, J. Kleinberg, E. Tardos, *Influential nodes in a diffusion model for social networks*, in Proceedings of the 32th International Colloquium on Automata, Languages and Programming, pages 1127-1138, 2005.

[18] F. Luccio, *Almost exact minimum feedback vertex set in meshes and butterflies*, Information Processing Letters 66 (1998) 59-64.

[19] F. Luccio, L. Pagli, H. Sanossian. *Irreversible dynamos in butterflies*, 6th Int. Coll. on Structural Information and Communication Complexity (SIROCCO), pages 204-218, 1999.

[20] D. Peleg, *Size bounds for dynamic monopolies*, Discrete Applied Mathematics 86 (1998) 263-273.

[21] D. Peleg, *Local majorities, coalitions and monopolies in graphs: a review*, Theoretical Computer Science 282 (2002) 231-257.

[22] M. Richardson, P. Domingos, *Mining knowledge-sharing sites for viral marketing*, in Proceedings of the 8th ACM SIGKDD, Edmonton, Canada, pages 61-70, 2002.

[23] F. S. Roberts, *Challenges for discrete mathematics and theoretical computer science in the defense against bioterrorism*, in Bioterrorism: Mathematical Modeling Applications in Homeland Security, Frontiers Appl. Mathem. 28, H. T. Banks and C. Castillo-Chavez, eds., SIAM, Philadelphia, pages 1-34, 2003.

[24] F. S. Roberts, *Graph-theoretical problems arising from defending against bioterrorism and controlling the spread of fires*, in Proceedings of the DIMACS/DIMATIA/Renyi Combinatorial Challenges Conference, Piscataway, NJ, 2006.

[25] S. Stueckle, R. D. Ringeisen, *Generalized Petersen graphs which are cycle permutation graphs*, Journal of Combinatorial Theory B, 37 (1984) 142-150.

[26] M. Watkins, *A Theorem on Tait Colourings with an Application to the Generalized Petersen Graphs*, Journal of Combinatorial Theory, 6 (1969) 152-164.

Appendix: [Not for publication - for referees' information only]

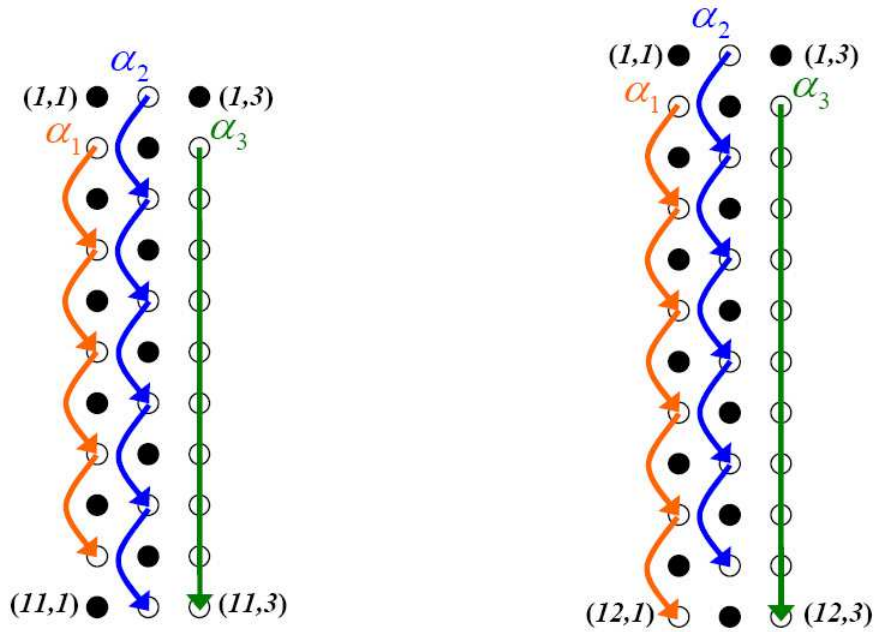

Figure 1. min-seed$(C_{11} \oslash C_3, 3) = 12$ (left) and min-seed$(C_{12} \oslash C_3, 3) = 13$ (right), where the target set S is the set of all black vertices, and the convinced sequence $\alpha_1 \sqcup \alpha_2 \sqcup \alpha_3$ is illustrated by three colored directed paths.

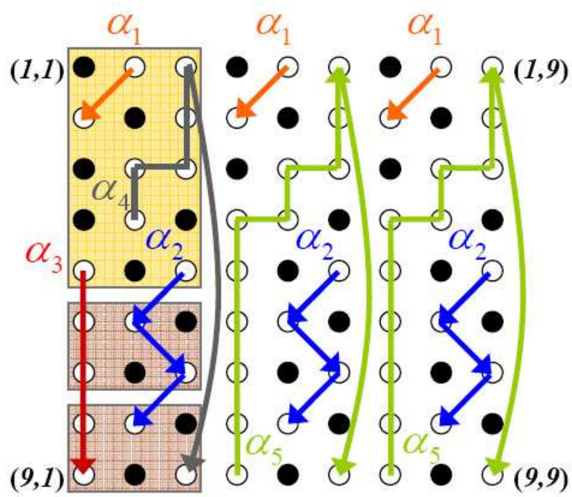

Figure 2. min-seed$(C_9 \oslash C_9, 3) = 28$.

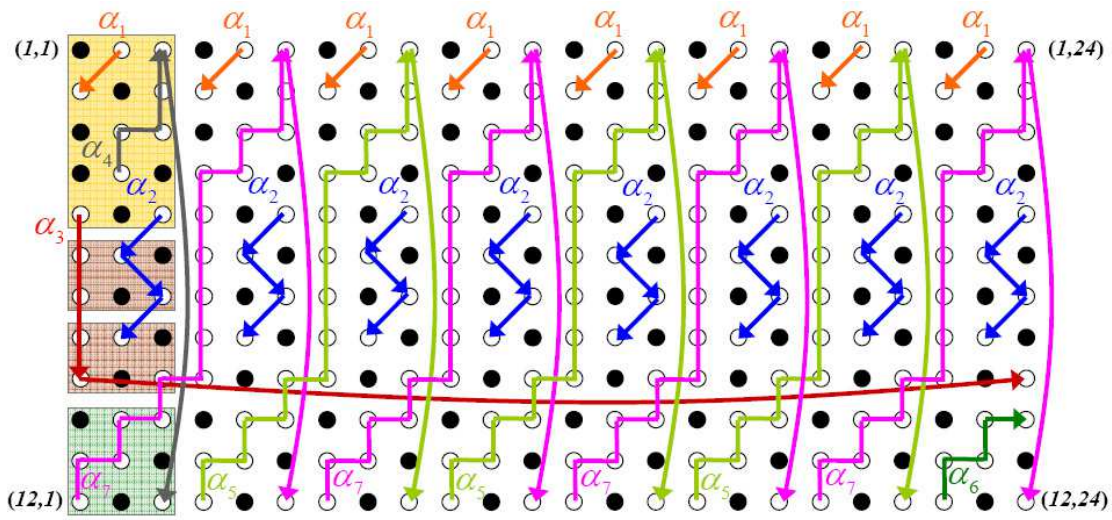

Figure 3. min-seed$(C_{12} \oslash C_{24}, 3) = 97$.

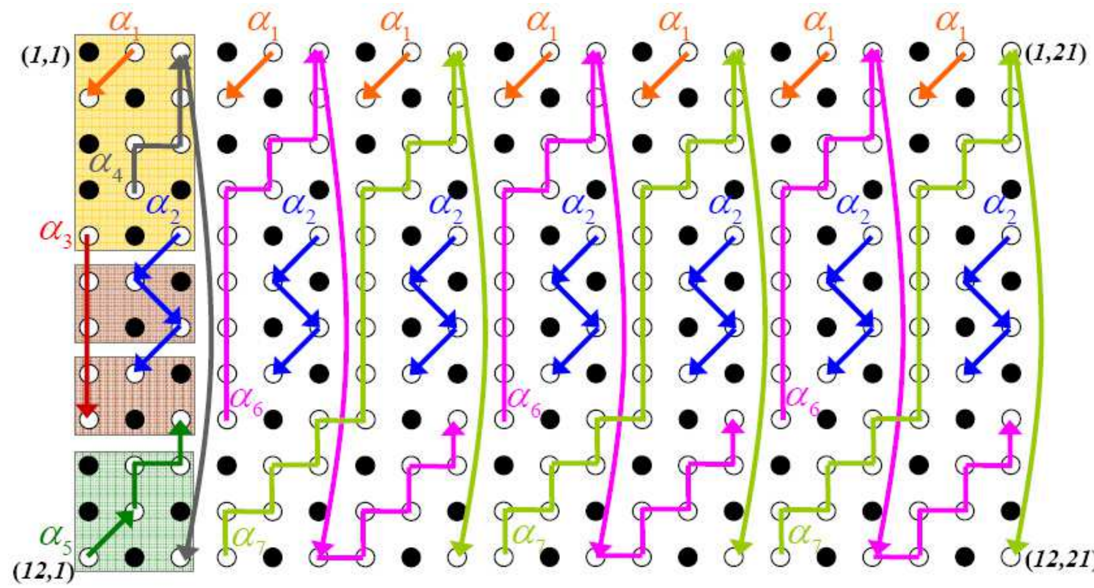

Figure 4. min-seed$(C_{12} \oslash C_{21}, 3) \leq 86$.

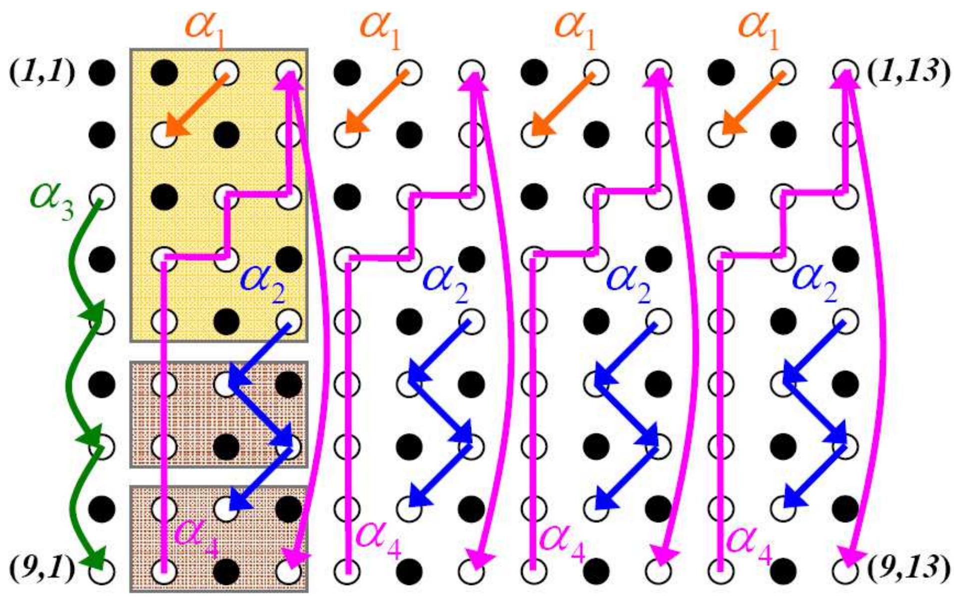

Figure 5. min-seed$(C_9 \oslash C_{13}, 3) \leq 41$.

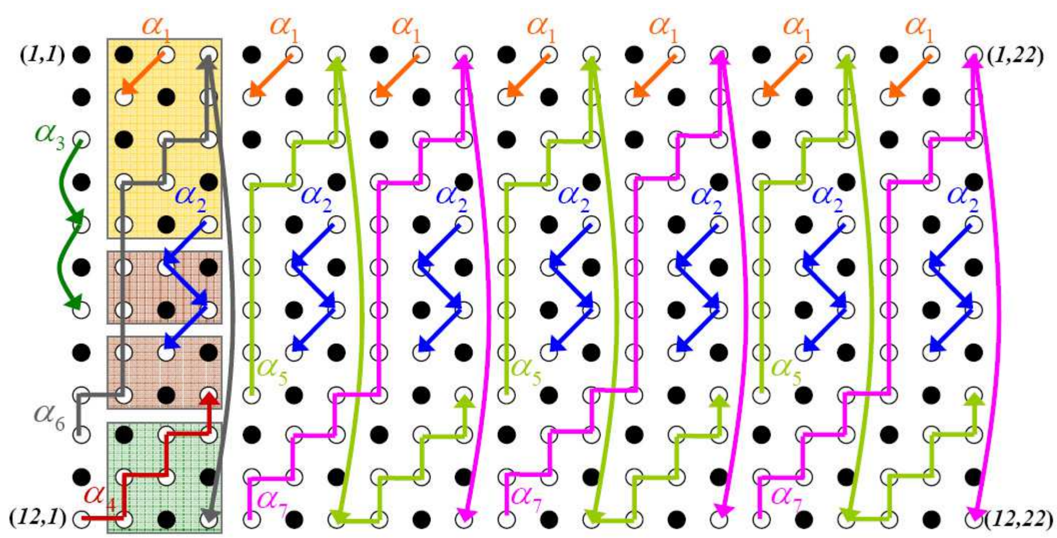

Figure 6. min-seed$(C_{12} \oslash C_{22}, 3) \leq 90$.

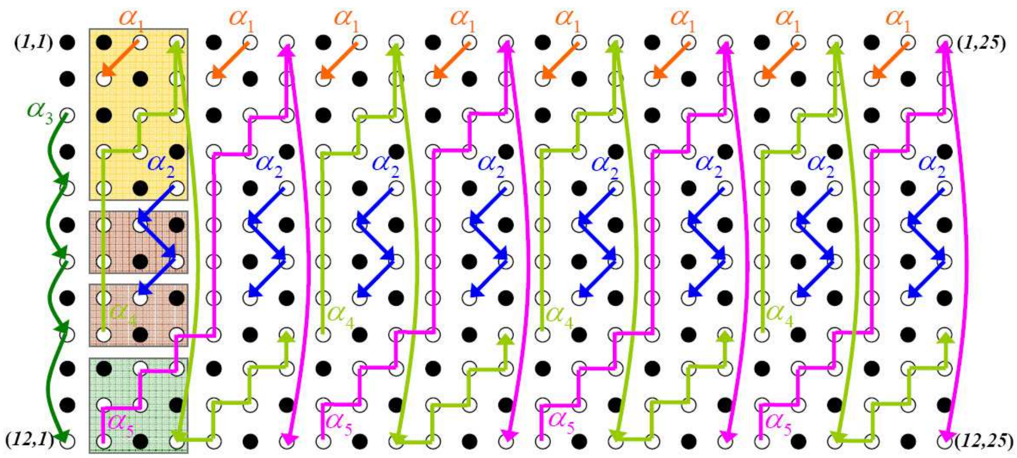

Figure 7. min-seed$(C_{12} \oslash C_{25}, 3) \leq 103$.

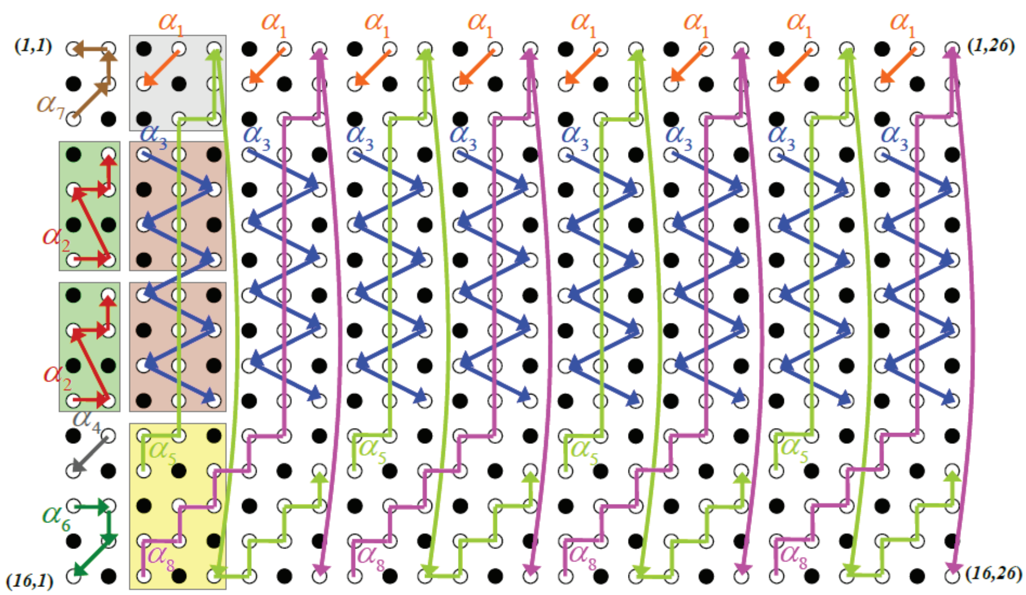

Figure 8. min-seed$(C_{16} \oslash C_{26}, 3) \leq 140$.

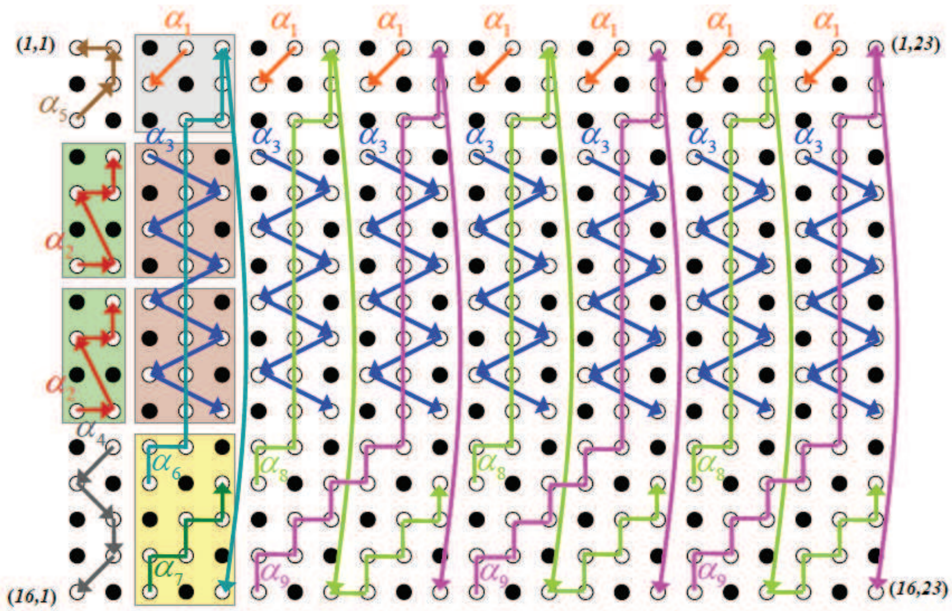

Figure 9. min-seed$(C_{16} \oslash C_{23}, 3) \leq 125$.

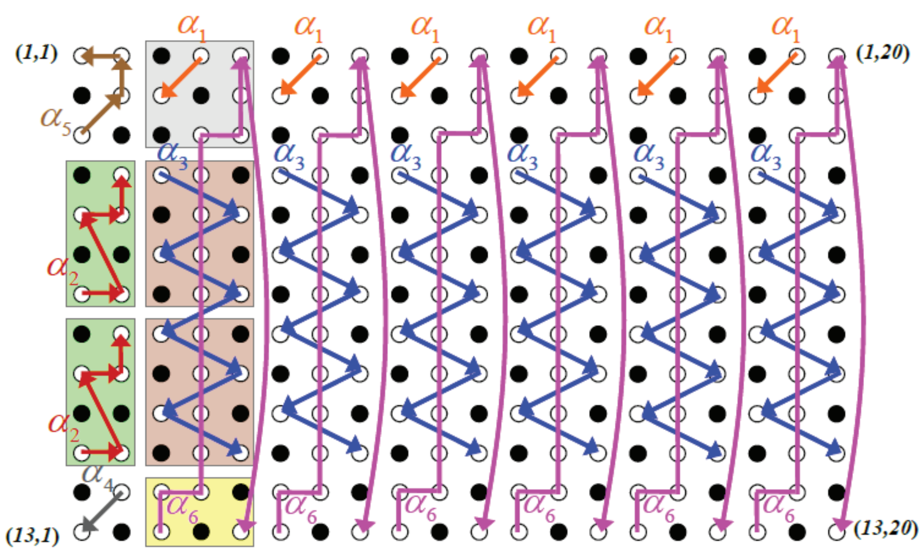

Figure 10. min-seed$(C_{13} \oslash C_{20}, 3) \leq 88$.

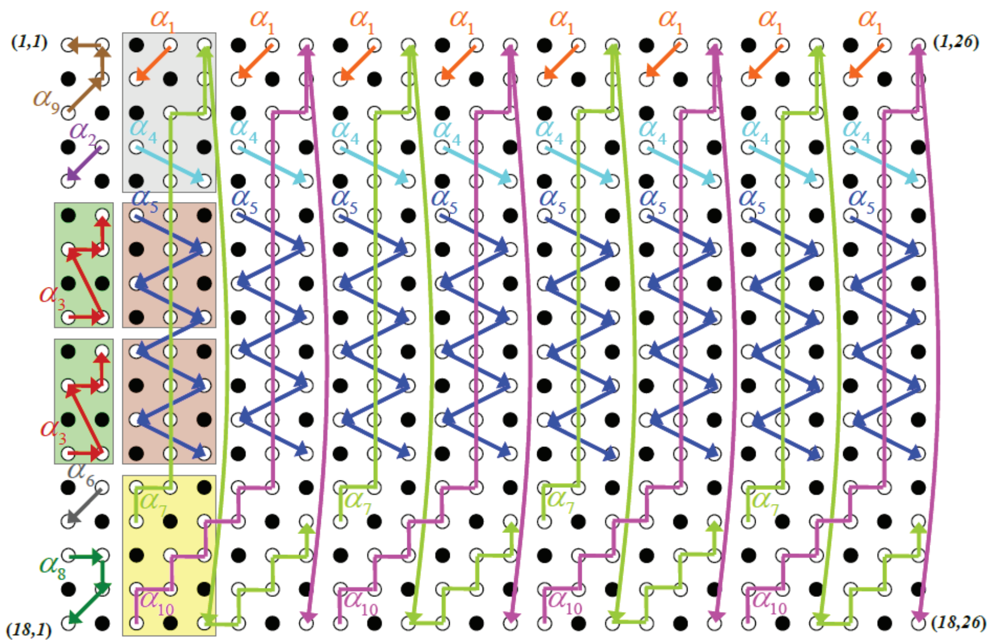

Figure 11. min-seed$(C_{18} \oslash C_{26}, 3) \leq 158$.

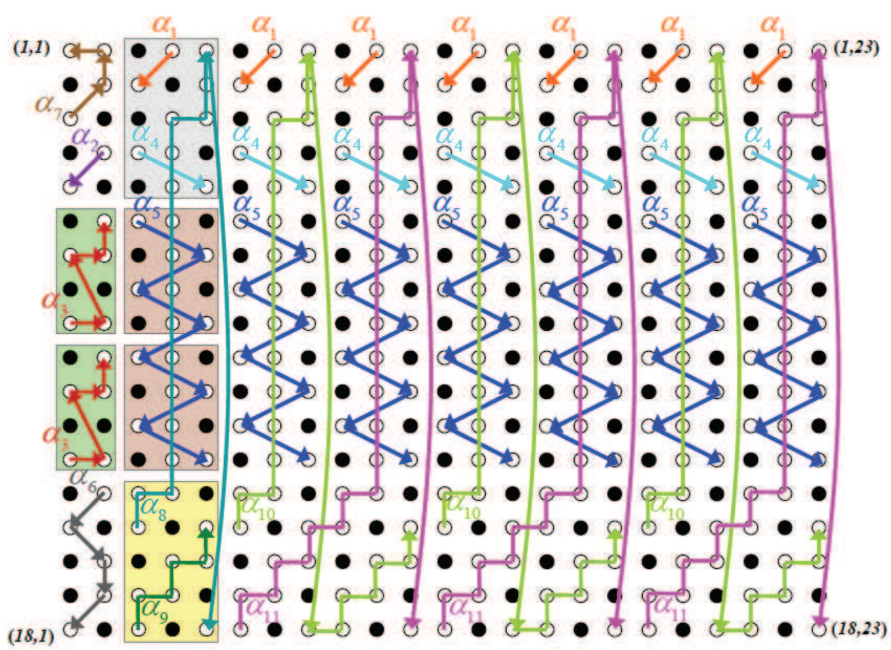

Figure 12. min-seed$(C_{18} \oslash C_{23}, 3) \leq 141$.

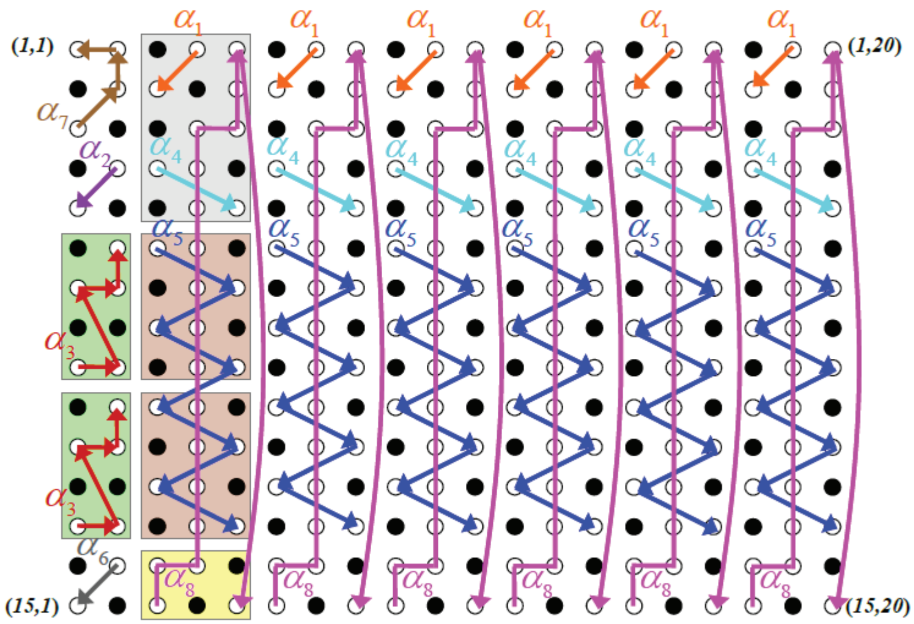

Figure 13. min-seed$(C_{15} \oslash C_{20}, 3) \leq 102$.

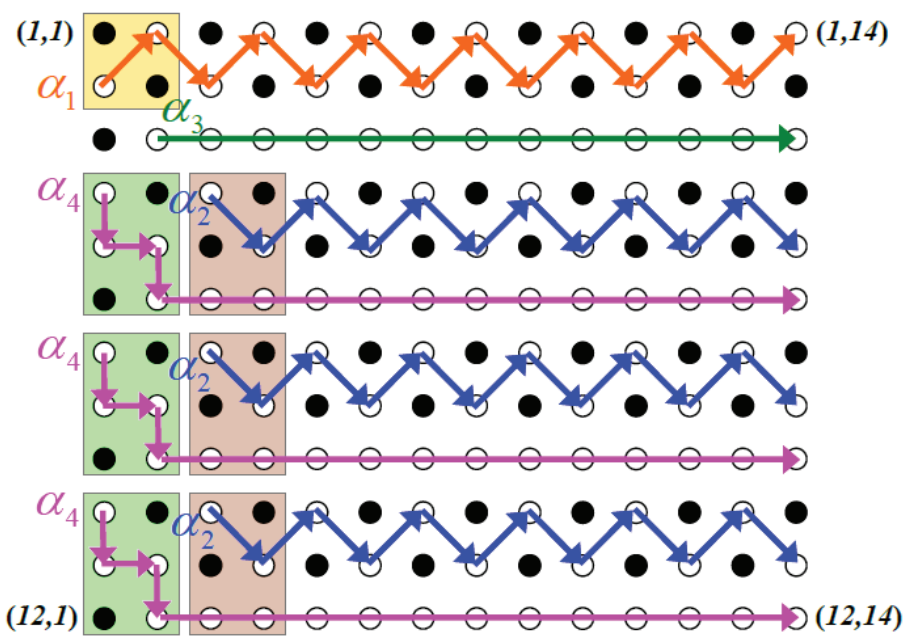

Figure 14. min-seed$(C_{12} \oslash C_{14}, 3) = 57$.

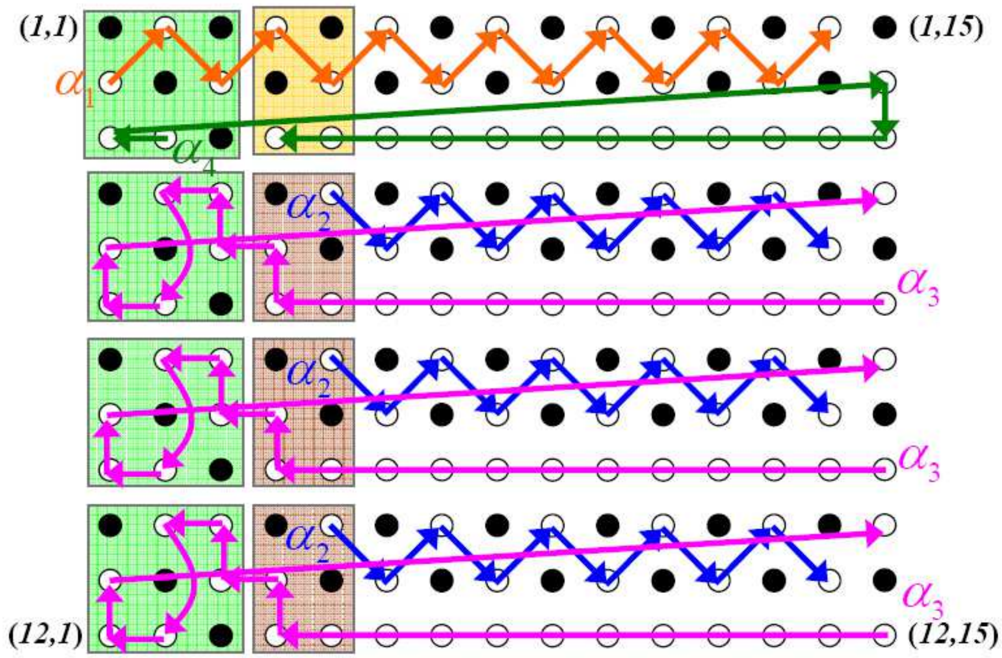

Figure 15. min-seed$(C_{12} \oslash C_{15}, 3) = 61$.